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Endogenous Economic Growth with Education Subsidies

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Abstract

The purpose of this study is to examine existence of business fluctuations in a growth model of endogenous wealth and human capital with government's subsidy policies on education by Zhang (2016). Zhang synthesized the Solow growth model and the Uzawa-Lucas two-sector model and took account of three ways of accumulating human capital: learning by producing, learning by education, and learning by consuming. This paper generalizes Zhang's model by allowing all the time-independent parameters to be time-dependent. It examines the relationship between growth and taxation with different time-dependent exogenous shocks. We simulate the model to demonstrate existence of equilibrium points, motion of the dynamic system, and oscillations due to different exogenous shocks.

Keywords:
Periodic shocks
Business cycles
Learning by consuming
Learning by education
Government subsidy
Propensity to receive education.

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1. Introduction

This study is to identify business cycles and economic fluctuations due to different exogenous shocks in a growth model with endogenous human capital and wealth. There are many studies on existence of business cycles (Chiarella & Flaschel, 2000; Gandolfo, 2005; Lorenz, 1993; Puu, 2011; Shone, 2002; Zhange, 2006; Zhangg, 2005; Zhangs, 1991). But there are only a few theoretical models which identify fluctuations due to dynamic interdependence between economic growth, wealth accumulation, human capital accumulation with education, learning by doing, and learning by consuming. This study attempts to provide another contribution to the literature by identifying economic fluctuations in an economic growth model of endogenous growth model with education subsidy proposed by Zhang (2016).

Human capital is generally considered as a key determinant of economic growth. Dynamic interdependence between economic growth and human capital is currently a main topic in economic theory. As education is an important way of accumulating human capital, many models have been proposed to examine interdependence between education and economic growth. As far as formal modeling of education and economic growth is concerned, the work by Lucas (1988) has caused a great interest in the issue among economists. In fact, the first formal dynamic growth model with education was proposed by Uzawa (1965). The Uzawa-Lucas model has been extended and generalized in various directions. Zhang (2016) introduces government subsidy on education into the Uzawa-Lucas two sector model. The sources of human capital accumulation are via three ways: Arrow's learning by doing, Uzawa's learning by education, and Zhang's creative leisure within a general equilibrium framework. The study extends Zhang's model to allow different exogenous time-dependent shocks. We demonstrate business cycles in Zhang's model due to exogenous periodic shocks. The rest paper is organized as follows. First we define the basic model. Then we show how we solve the dynamics and simulates the model. Thirdly we examine effects of changes in some parameters on the economic system over time. Finally we conclude the study. The appendix proves the procedure in the lemma.

2. The Basic Model

This section is built on Zhang (2016) by allowing all the time-dependent parameters to be time-dependent in the model with one production sector and one education sector. The economy has. Most aspects of the production sector are similar to the standard one-sector growth model (Solow, 1956). It is assumed that there is only one (durable) good in the economy under consideration. We select the commodity to serve as numeraire, with all the other prices being measured relative to its price. Households own assets of the economy and distribute their incomes to consumption, education and wealth accumulation. The production sectors or firms use physical

capital and labor as inputs. Exchanges take place in perfectly competitive markets. Factor markets work well; factors are inelastically supplied and the available factors are fully utilized at every moment. Saving is undertaken only by households. All earnings of firms are distributed in the form of payments to factors of production. We assume a homogenous and fixed population $N_0(t)$. Let $T(t)$ and $T_e(t)$ represent for, respectively, the work time and study time of a representative household. The total work time, $N(t)$, is $T(t)N_0(t)$. The total capital stock of physical capital, $K(t)$, is allocated between the two sectors. We use $N_e(t)$ and $K_e(t)$ to stand for the labor force and capital stocks employed by the education sector, and $N_i(t)$ and $K_i(t)$ for the labor force and capital stocks employed by the production sector. As labor and capital are assumed fully employed, we have
$K_i(t) + K_e(t) = K(t), \quad N_i(t) + N_e(t) = N(t).$

We rewrite the above relations as follows

$$n_i(t)k_i(t) + n_e(t)k_e(t) = k(t), \quad n_i(t) + n_e(t) = 1, \tag{1}$$

in which

$$k_j(t) \equiv \frac{K_j(t)}{N_j(t)}, \quad n_j(t) \equiv \frac{N_j(t)}{N(t)}, \quad k(t) \equiv \frac{K(t)}{N(t)}, \quad j = i, e.$$

2.1. The Production Sector

We use $H(t)$ to present for the level of human capital of the population. We assume that production is to combine 'qualified labor force', $H^m(t)N_i(t)$, and physical capital, $K_i(t)$, where the time-dependent exogenous variable $m(t)$ describes the efficiency of how effectively the population uses human capital. We use the conventional production function to describe a relationship between inputs and output. The production function $F_i(t)$ is specified as follows

$$F_i(t) = A_i(t)K_i^{\alpha_i(t)}(t)\left(H^{m(t)}(t)N_i(t)\right)^{\beta_i(t)}, \quad A_i(t), \alpha_i(t), \beta_i(t) > 0, \quad \alpha_i(t) + \beta_i(t) = 1,$$

where $A_i(t), \alpha_i(t),$ and $\beta_i(t)$ are positive parameters. The rate of interest $r(t)$ and wage rate $w(t)$ are determined by markets. The marginal conditions are given by

$$\begin{aligned} r(t) + \delta_k(t) &= \frac{\alpha_i(t)\bar{\tau}_i(t)F_i(t)}{K_i(t)} = \bar{\tau}_i(t)\alpha_i(t)A_i(t)H^{m(t)\beta_i(t)}k_i^{-\beta_i(t)}(t), \\ w(t) &= \frac{\beta_i(t)\bar{\tau}_i(t)F_i(t)}{N_i(t)} = \bar{\tau}_i(t)\beta_i(t)A_i(t)H^{m(t)\beta_i(t)}k_i^{\alpha_i(t)}(t), \end{aligned} \tag{2}$$

Where $\delta_k(t)$ is depreciation rate of physical capital and $\bar{\tau}_i(t) \equiv 1 - \tau_i(t)$, $\tau_i(t)$ being the fixed tax rate on the industrial output.

2.2. Accumulation of Human Capital and the Education Sector

We assume that there are three sources of improving human capital, through education (Uzawa, 1965) learning by producing (Arrow, 1962) and learning by leisure (Zhang, 2016). We propose that human capital dynamics is given by

$$\dot{H}(t) = \frac{\upsilon_e(t)F_e^{a_e(t)}(t)\left(H^{m(t)}(t)T_e(t)N_0(t)\right)^{b_e(t)}}{H^{\pi_e(t)}N_0(t)} + \frac{\upsilon_i(t)F_i^{a_i}(t)}{H^{\pi_i(t)}N_0(t)} + \frac{\upsilon_h(t)C^{a_h(t)}}{H^{\pi_h(t)}(t)N_0(t)} - \delta_h(t)H(t), \tag{3}$$

where $\delta_h(t) (> 0)$ is the depreciation rate of human capital, $\upsilon_e(t), \upsilon_i(t), \upsilon_h(t), a_e(t),$ $b_e(t),$ $a_i(t)$, and $a_h(t)$ are non-negative parameters. The signs of the parameters, $\pi_e(t),$ $\pi_i(t),$ and $\pi_h(t)$ are not specified as they can be either negative or positive.

The education sector is characterized of perfect competition. The education sector charges students $p(t)$ per unit of time. The education sector pays teachers and capital with the market rates. The cost of the education sector is given by $w(t)N_e(t) + r(t)K_e(t)$. The total education service is measured by the total education time received by the population, $T_e(t)N_0(t)$. The production function of the education sector is assumed to be a function of $K_e(t)$ and $N_e(t)$. We specify the production function of the education sector as follows

$$F_e(t) = A_e(t)K_e^{\alpha_e(t)}(t)\left(H^{m(t)}(t)N_e(t)\right)^{\beta_e(t)}, \quad \alpha_e(t), \beta_e(t) > 0, \quad \alpha_e(t) + \beta_e(t) = 1, \tag{4}$$

where $A_e(t)$, $\alpha_e(t)$ and $\beta_e(t)$ are positive parameters. The education sector maximizes the following profit

$$\pi(t) = \bar{\tau}_e(t) p(t) A_e(t) K_e^{\alpha_e(t)} \left(H^{m(t)}(t) N_e(t)\right)^{\beta_e(t)} - \left(r(t) + \delta_k(t)\right) K_e(t) - w(t) N_e(t),$$

where $\bar{\tau}_e(t) \equiv 1 - \tau_e(t)$, and $\tau_e(t)$ is the fixed tax rate on the education service. The optimal solution is given by

$$r(t) + \delta_k(t) = \alpha_e(t) \bar{\tau}_e(t) A_e(t) p(t) H(t)^{m(t)\beta_e(t)} k_e^{-\beta_e(t)}(t),$$
$$w(t) = \beta_e(t) \bar{\tau}_e(t) A_e(t) p(t) H^{m(t)\beta_e(t)} k_e^{\alpha_e(t)}(t). \tag{5}$$

2.3. Consumer Behaviors

We denote per capita wealth by $\bar{k}(t)$, where $\bar{k}(t) \equiv K(t)/N_0(t)$. We use $\tau_h(t)$, $\tau_w(t)$ and $\tau_c(t)$ to stand for the fixed tax rates on the wealth income, wage income and consumption. Per capita current income from the interest payment $\bar{\tau}_h r(t)\bar{k}(t)$ and the wage payment $\bar{\tau}_w T(t) w(t)$ is given by

$$y(t) = \bar{\tau}_h(t) r(t) \bar{k}(t) + \bar{\tau}_w(t) T(t) w(t),$$

where $\bar{\tau}_h(t) \equiv 1 - \tau_h(t)$ and $\bar{\tau}_w(t) \equiv 1 - \tau_w(t)$. The per capita disposable income is given by

$$\hat{y}(t) = y(t) + \bar{k}(t) = \left(1 + \bar{\tau}_h(t) r(t)\right)\bar{k}(t) + \bar{\tau}_w(t) T(t) w(t). \tag{6}$$

Let $\tau(t)$ stand for the subsidy per unit of time people receive from the government for education. The education cost is the price minus the subsidy rate, $p(t) - \tau(t)$. At each point of time, a consumer would distribute the total available budget between saving $s(t)$, consumption of goods $c(t)$, and education $T_e(t)$. The budget constraint is given by

$$\bar{\tau}_c c(t) + s(t) + \left(p(t) - \tau(t)\right) T_e(t) = \left(1 + \bar{\tau}_h r(t)\right)\bar{k}(t) + \bar{\tau}_w T(t) w(t), \tag{7}$$

where $\bar{\tau}_c(t) \equiv 1 + \tau_c(t)$. The consumer is faced with the following time constraint

$$T(t) + T_e(t) = T_0,$$

where T_0 is the total available time for work and study. Substituting this function into the budget constraint (7) yields

$$\bar{\tau}_c(t) c(t) + s(t) + \bar{p}(t) T_e(t) = \bar{y}(t) \equiv \left(1 + \bar{\tau}_h(t) r(t)\right)\bar{k}(t) + \bar{\tau}_w(t) T_0 w(t), \tag{8}$$

where $\bar{p}(t) \equiv p(t) + w(t) - \tau(t)$.

We assume that consumers' utility function is a function of $c(t)$, $s(t)$, and $T_e(t)$, as follows

$$U(t) = c^{\xi_0(t)}(t) s^{\lambda_0(t)}(t) T_e^{\eta_0(t)}(t), \tag{9}$$

where $\xi_0(t)$ is called the propensity to consume, $\lambda_0(t)$ the propensity to own wealth, and $\eta_0(t)$ the propensity to obtain education. For the representative consumer $w(t)$, and $r(t)$ are given in markets. Maximizing $U(t)$ subject to (8) yields

$$c(t) = \xi(t) \bar{y}(t), \quad s(t) = \lambda(t) \bar{y}(t), \quad \bar{p}(t) T_e(t) = \eta(t) \bar{y}, \tag{10}$$

where

$$\xi(t) \equiv \frac{\rho(t) \xi_0(t)}{\bar{\tau}_c(t)}, \quad \lambda(t) \equiv \rho(t) \lambda_0(t), \quad \eta(t) \equiv \rho(t) \eta_0(t), \quad \rho(t) = \frac{1}{\xi_0(t) + \lambda_0(t) + \eta_0(t)}.$$

We now find dynamics of capital accumulation. According to the definition of $s(t)$, the change in the household's wealth is given by

$$\dot{\bar{k}}(t) = s(t) - \bar{k}(t) = \lambda(t) \bar{y}(t) - \bar{k}(t). \tag{11}$$

For the education sector, the demand and supply balances at any point of time

$$T_e(t) N_0(t) = F_e(t). \tag{12}$$

As the government's tax is spent only on subsidizing education, we have

$$\tau(t) T_e(t) = \frac{\tau_i(t) F_i(t)}{N_0(t)} + \frac{\tau_e(t) p(t) F_e(t)}{N_0(t)} + \tau_c(t) c(t) + \tau_h(t) r(t) \bar{k}(t) + \tau_w(t) T(t) w(t). \tag{13}$$

As output of the production sector is equal to the sum of the level of consumption, the depreciation of capital stock and the net savings, we have

$$C(t) + S(t) - K(t) + \delta_k(t) K(t) = F_i(t), \tag{14}$$

where $C(t)$ is the total consumption, $S(t) - K(t) + \delta_k(t) K(t)$ is the sum of the net saving and depreciation, that is

$$C(t) = c(t) N_0(t), \quad S(t) = s(t) N_0(t).$$

We have thus built the dynamic model. We now examine dynamics of the model.

3. The Dynamics and its Properties

This section examines dynamics of the model. First, we show that the dynamics can be expressed by the two-dimensional differential equations system with $k_i(t)$ and $H(t)$ as the variables. Lemma

The dynamics of the economic system is governed by the 2 -dimensional differential equations

$$\dot{k}_i(t) = \tilde{\Omega}_i(k_i(t), H(t), t),$$
$$\dot{H}(t) = \tilde{\Omega}_h(k_i(t), H(t), t), \quad (15)$$

where the functions $\tilde{\Omega}_i$ and $\tilde{\Omega}_h$ are functions of $k_i(t)$, $H(t)$, and t, defined in the Appendix. Moreover, all the other variables can be determined as functions of $k_i(t)$ and $H(t)$ at any point of time by the following procedure: $k(t)$ by (A9) $\rightarrow$ $T(t)$ by (A8) $\rightarrow$ $T_e(t) = T_0 - T(t)$ $\rightarrow$ $\bar{k}(t) = k(t)T(t)$ $\rightarrow$ $k_h(t) = \alpha k_i(t)$ $\rightarrow$ $\bar{y}(t)$ by (A10) $\rightarrow$ $p(t)$ by (A2) $\rightarrow$ $n_i(t)$ and $n_h(t)$ by (A3) $\rightarrow$ $r(t)$ and $w(t)$ by (2) $\rightarrow$ $c(t)$ and $s(t)$ by (10) $\rightarrow$ $N(t) = N_0 T(t)$ $\rightarrow$ $N_j(t) = n_j(t)N(t)$, $j = i, e$ $\rightarrow$ $K(t) = k(t)N(t)$ $\rightarrow$ $K_j(t) = k_j(t)N_j(t)$ $\rightarrow F_j(K_j(t), N_j(t))$ $\rightarrow$ $\tau(t)$ and $s(t)$ by (13).

In the reminder of this section we illustrate the results by Zhang (2016) when all the parameters are constant. In the next section we simulate the model when we allow parameters to be exogenously time-dependent. We specify the depreciation rates by $\delta_k = 0.05$, $\delta_h = 0.04$, and let $T_0 = 1$. We specify the other parameters as follows

$$\alpha_i = 0.34,\ \alpha_e = 0.55,\ \lambda_0 = 0.6,\ \xi_0 = 0.08,\ \eta_0 = 0.007,\ N_0 = 50000,\ A_i = 0.9,$$
$$A_e = 0.7,\ m = 0.7,\ v_e = 0.8,\ v_i = 2.5,\ v_h = 0.7,\ a_e = 0.3,\ b_e = 0.5,\ a_i = 0.4, \quad (16)$$
$$a_h = 0.1,\ b_h = 0.3,\ \pi_e = -0.1,\ \pi_i = 0.7,\ \pi_h = 0.1,\ \tau_i = \tau_w = \tau_h = \tau_e = \tau_c = 0.005.$$

By (14), an equilibrium point of the dynamic system is given by

$$\Omega(k_i, H) = 0,\ \tilde{\Omega}(k_i, H) = 0.$$

Simulation demonstrates that the above equations have the following unique equilibrium solution

$$k_i = 5.98,\ H = 0.33.$$

The equilibrium values of the other variables are given by the procedure in the lemma. We list them as follows

$$H = 0.33,\ N = 47047.8,\ N_i = 45651.5,\ N_e = 1396.26,\ K_i = 273171,\ K_e = 19822.7,$$
$$f_i = 0.98,\ f_e = 1.32,\ F_i = 44931.7,\ F_e = 1835.6,\ k = 6.23,\ k_i = 5.98,\ k_e = 14.20,$$
$$p = 0.68,\ w = 0.65,\ \tau = 0.20,\ \bar{k} = 5.86,\ T = 0.94,\ T_e = 0.06,\ c = 0.78.$$

It is straightforward to calculate the two eigenvalues as follows: -0.121 and -0.04. As the two eigenvalues are negative, the unique equilibrium is locally stable. We specify initial conditions as follows: $k_i(0) = 6.2$ and $H(0) = 0.3$. The simulation result is plotted in Figure 1. The level of the human capital increases from the initial state to the equilibrium value. The wealth and consumption experience a kind of J – curve process. It first experiences declination in per capita levels of consumption and wealth. After some time these variables start to increase. The education time slightly declines and soon begins to increase. During the simulation period, the education fee and subsidizing fee increase.

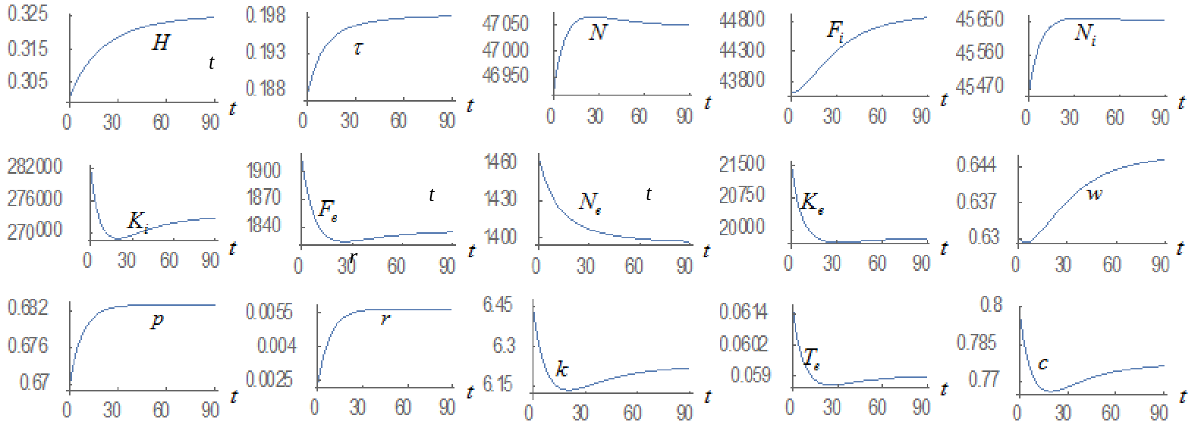

Figure-1. The Motion of the Economic System.

4. Comparative Dynamic Analysis

We now examine impact of periodic changes in different exogenous factors on the dynamic processes of the system. We introduce variable $\overline{\Delta} x(t)$ to stand for the change rate of the variable, $x(t)$, in percentage due to changes in the parameter value.

4.1. Fluctuations in the Total Productivity of the Education Sector

First, we examine the case that the total productivity of the education sector is oscillatory as follows: $A_e(t) = 0.7 + 0.05\sin(t)$. The simulation results are plotted in Figure 2.

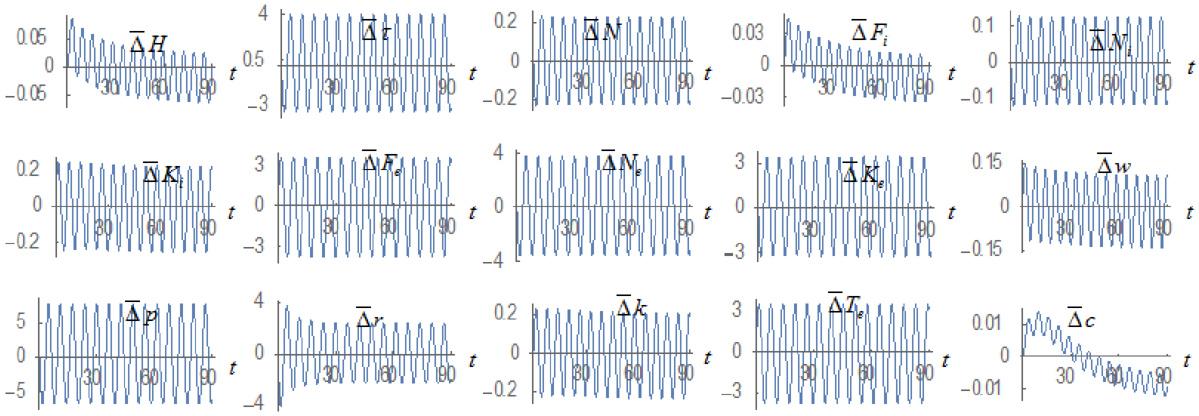

Figure-2. Fluctuations in the Total Productivity of the Education Sector.

4.2. Fluctuations in the Tax Rate on Output of the Industrial Sector

We increase the tax rate on the output level of the industrial sector in the following way: $\tau_i(t) = 0.005 + 0.01\sin(t)$. The simulation results are plotted in Figure 3.

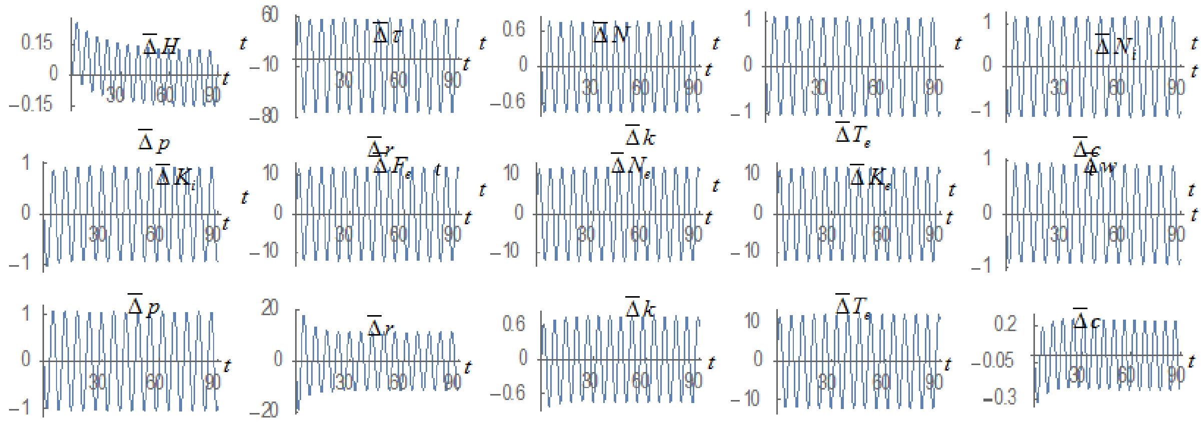

Figure-3. Fluctuations in the Tax Rate on Output of the Industrial Sector.

4.3. Fluctuations in the Propensity to Obtain Education

It is important to examine effects of changes in the household's preference for education. We allow the propensity to receive education to increase in the following way: $\eta_0(t) = 0.007 + 0.003\sin(t)$. The simulation results are demonstrated in Figure 4.

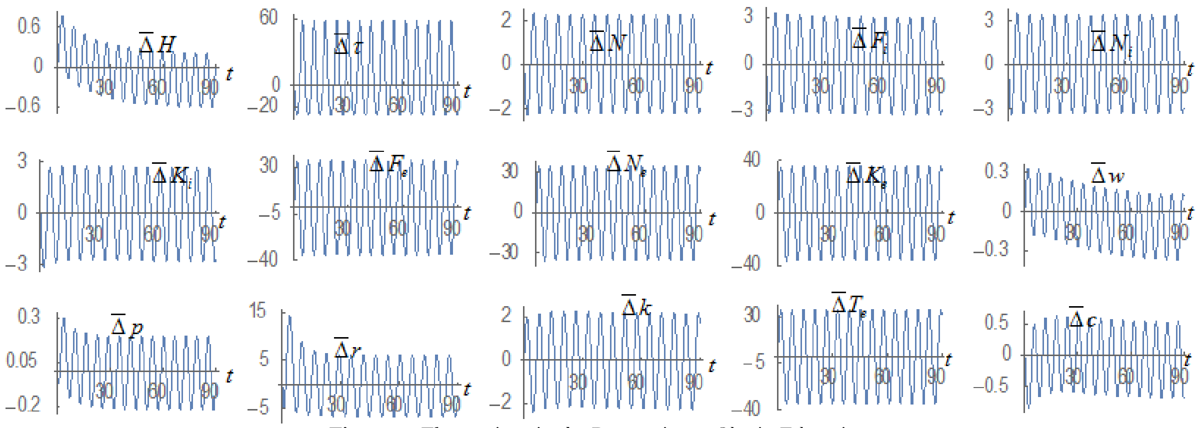

Figure-4. Fluctuations in the Propensity to Obtain Education.

5. Concluding Remarks

This paper showed economic oscillations due to periodic changes in some parameters in the economic model proposed by Zhang (2016). Zhang built a two-sector growth model with wealth accumulation and human capital accumulation. It emphasized the role of government's subsidy policies on economic growth and human capital accumulation. The study made a unique approach to the issue in that it treated the sources of human capital via three ways: Arrow's learning by doing, Uzawa's learning by education, and Zhang's creative leisure within a general equilibrium framework. Although Zhang conducted statics analysis, this study generalized Zhang's model and conducted comparative dynamic analysis. We simulated the model to demonstrate existence of equilibrium points, motion of the dynamic system, and oscillations due to different exogenous shocks.

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Appendix: Proving the Lemma

From (2) and (5), we obtain

$$\frac{K_e}{N_e} = \alpha \frac{K_i}{N_i}, \quad \text{i.e.,} \quad k_e = \alpha k_i, \tag{A1}$$

where $\alpha \equiv \alpha_e \beta_i / \alpha_i \beta_e$ ($\neq 1$ assumed). From (A1), (2) and (4), we obtain

$$p(t) = \frac{\bar{\tau}_i \alpha_i A_i}{\bar{\tau}_e \alpha_e A_e} \alpha^{\beta_e} H^{-m\beta} k_i^{\beta}, \tag{A2}$$

where $\beta \equiv \beta_e - \beta_i$. From (A1) and (1), we solve the labor distribution as functions of $k_i(t)$ and $k(t)$

$$n_i = \frac{\alpha k_i - k}{(\alpha - 1) k_i}, \quad n_e = \frac{k - k_i}{(\alpha - 1) k_i}. \tag{A3}$$

Dividing (14) by N_0, we have

$$c + s - \delta \bar{k} = A_i T n_i H^{\beta_i m} k_i^{\alpha_i},$$

where $\delta \equiv 1 - \delta_k$. Substituting $c = \xi \bar{y}$ and $s = \lambda \bar{y}$ into the above equation yields

$$\bar{y}=\left\{\frac{\alpha A_i H^{\beta_i m} k_i^{\alpha_i}}{\alpha-1}+\delta k-\frac{A_i H^{\beta_i m} k}{(\alpha-1)k_i^{\beta_i}}\right\}\frac{T}{\xi+\lambda}, \tag{A4}$$

where we use the equation for n_i in (A3) and $\bar{k}=Tk$. Insert (2) and $\bar{k}=Tk$ into the definition of $\bar{y}$ in (8)

$$\bar{y}=\left(1-\bar{\tau}_h\delta_k+\bar{\tau}_h\bar{\tau}_i\alpha_i A_i H^{m\beta_i}k_i^{-\beta_i}\right)kT+\bar{\tau}_w\bar{\tau}_i T_0\beta_i A_i H^{m\beta_i}k_i^{\alpha_i}. \tag{A5}$$

From (A4) and (A5), we solve

$$\left(\alpha A_\xi H^{\beta_i m}k_i^{\alpha_i}+\bar{\delta}k-\frac{A_\xi H^{\beta_i m}k}{k_i^{\beta_i}}-\bar{\tau}_h\bar{\tau}_i\alpha_i A_i H^{m\beta_i}k_i^{-\beta_i}k\right)T=\bar{\tau}_w\bar{\tau}_i T_0\beta_i A_i H^{m\beta_i}k_i^{\alpha_i}, \tag{A6}$$

where

$$\bar{\delta}\equiv\frac{\delta}{\xi+\lambda}-\left(1-\bar{\tau}_h\delta_k\right),\quad A_\xi\equiv\frac{A_i}{(\alpha-1)(\xi+\lambda)}.$$

From (12) and (4), we have

$$T_e=A_e T n_e H^{\beta_e m}k_e^{\alpha_e}. \tag{A7}$$

Insert $T+T_e=T_0$ and n_e in (A3) in (A7)

$$T=\left(1+\frac{\alpha^{\alpha_e}A_e H^{\beta_e m}\left(k-k_i\right)}{(\alpha-1)k_i^{\beta_e}}\right)^{-1}T_0. \tag{A8}$$

Substituting (A8) into (A6) yields

$$k=\varphi(k_i,H)\equiv\frac{\left(1-\alpha^{\alpha_e}A_e k_i^{\alpha_e}H^{\beta_e m}/(\alpha-1)\right)\bar{\tau}_w\bar{\tau}_i\beta_i A_i k_i-\alpha A_\xi k_i}{\bar{\delta}k_i^{\beta_i}H^{-\beta_i m}-A_\xi-\bar{\tau}_h\bar{\tau}_i\alpha_i A_i-\alpha^{\alpha_e}\bar{\tau}_w\bar{\tau}_i\beta_i A_i A_e k_i^{\alpha_e}H^{\beta_e m}/(\alpha-1)}. \tag{A9}$$

By (A9), we can express $k(t)$ as functions of $k_i(t)$ and $H(t)$ at any point of time. By (A8) and (A5), we can also express $T(t)$ and $\bar{y}(t)$ as functions of $k_i(t)$ and $H(t)$ as follows

$$T=\varphi_0(k_i,H),\quad \bar{y}=\Lambda(k_i,H).$$

With (3), it is straightforward to show that the motion of human capital can be expressed as a function of $k_i(t)$ and $H(t)$ at any point of time

$$\dot{H}(t)=\tilde{\Omega}_h\left(k_i(t),H(t)\right). \tag{A10}$$

We now show that change in $k_i(t)$ can also be expressed as a differential equation in terms of $k_i(t)$ and $H(t)$. First, substitute $\bar{y}=\Lambda$ and $\bar{k}=Tk=\varphi_0\varphi$ into (11)

$$\dot{\bar{k}}(t)=\lambda\Lambda\left(k_i(t),H(t)\right)-\varphi_0\left(k_i(t),H(t)\right)\varphi\left(k_i(t),H(t)\right). \tag{A11}$$

Taking derivatives of $\bar{k}=Tk=\varphi_0\varphi$ with respect to time, we have

$$\dot{\bar{k}}=\left(\frac{\partial\varphi_0}{\partial k_i}\varphi+\frac{\partial\varphi}{\partial k_i}\varphi_0\right)\dot{k}_i+\left(\frac{\partial\varphi_0}{\partial H}\varphi+\frac{\partial\varphi}{\partial H}\varphi_0\right)\tilde{\Omega}_h+\frac{\partial\varphi_0}{\partial t}\varphi+\frac{\partial\varphi}{\partial t}\varphi_0, \tag{A12}$$

where we use (A10). Substituting (A12) into (A11) yields

$$\dot{k}_i=\tilde{\Omega}_i(k_i,H)\equiv\left[\lambda\Lambda-\varphi_0\varphi-\left(\frac{\partial\varphi_0}{\partial H}\varphi+\frac{\partial\varphi}{\partial H}\varphi_0\right)\tilde{\Omega}_h-\frac{\partial\varphi_0}{\partial t}\varphi-\frac{\partial\varphi}{\partial t}\varphi_0\right]\left(\frac{\partial\varphi_0}{\partial k_i}\varphi+\frac{\partial\varphi}{\partial k_i}\varphi_0\right)^{-1}. \tag{A13}$$

The two differential equations (A10) and (A13) contain two variables $k_i(t)$ and $H(t)$. We thus proved the lemma.