International Journal of Educational Technology and Learning
ISSN: 2523-0581
Vol. 17, No. 2, pp. 28-42, 2024
DOI: 10.55217/101.v17i2.845

Wolfram alpha and maple as educational technology for future teachers of mathematics

Sergei Abramovich

School of Education and Professional Studies, State University of New York at Potsdam, USA.
Email: abramovs@potsdam.edu

Abstract

The paper's purpose is to present technology-enhanced activities with triangular, square, and other polygonal numbers arranged in basic geometric shapes – equilateral and isosceles triangles and squares. The author's review of recently published mathematics education papers found no use of computing technology in teaching polygonal numbers either at the college or the pre-college levels worldwide. This omission served as a motivation to share and discuss computational algorithms designed by the author for the summation of such numbers within each geometric structure. A method through which those algorithms were used with teacher candidates is based on the technology-immune/technology-enabled problem-solving pedagogy. This method can be recommended for the modern-day teaching of elementary number theory and other mathematical topics across multiple grade levels and educational programs. The activities, supported by *Wolfram Alpha* and *Maple*, can be used by instructors of technology-motivated mathematics teacher education courses. The paper argues that the power of digital tools allows future teachers of mathematics, in the context of elementary number theory, to appreciate the use of simple algorithms in achieving sophisticated computational outcomes. Reflective comments of teacher candidates (the author's students) are shared to indicate the importance of the history of mathematics and the focus on mathematical visualization as a pedagogical approach aimed at conceptual understanding of the subject matter.

Keywords:
Number theory
Polygonal numbers
Teacher education
Technology.

Publisher:
Scientific Publishing Institute

Received: *30 August 2024*
Revised: *10 October 2024*
Accepted: *21 October 2024*
Published: *31 October 2024*

Funding: This study received no specific financial support.
Institutional Review Board Statement: Not applicable.
Transparency: The author confirms that the manuscript is an honest, accurate, and transparent account of the study; that no vital features of the study have been omitted; and that any discrepancies from the study as planned have been explained. This study followed all ethical practices during writing.
Competing Interests: The author declares that there are no conflicts of interests regarding the publication of this paper.

1. Introduction

It has been almost a century since Godfrey Harold Hardy, a prominent British number theorist, recommended topics of elementary number theory for early mathematical instruction (Hardy, 1929). In the digital era, this recommendation was reflected in similar statements by mathematicians and mathematics educators extending instruction from yearly grades to the entire K-16 context (Abramovich, Fujii, & Wilson, 1995; Koshy, 2002; Vavilov, 2020; Zazkis & Campbell, 2006). One number theory topic deals with polygonal (figurate) numbers. The representation of numbers as simple geometric figures goes back to mathematics of antiquity when certain combinations of integers were used to describe various shapes and shapes were found to portray both explicit and hidden properties of integers. The paper is written to show how a numeric approach to the concepts of elementary number theory made possible, as discussed in Abramovich et al. (1995) by spreadsheet modeling, can be extended to enable a symbolic approach supported by digital tools like *Wolfram Alpha* and *Maple*. This extension allows one using symbolic computations in the context of mathematics teacher education; that is, using simple coding without teaching complicated computational algorithms.

Wolfram Alpha and *Maple* are examples of digital tools capable of replacing traditional paper and pencil algebraic transformations such as, for example, finding the sum of the first n triangular numbers. In the past, mathematics educators used a numeric approach to finding this sum by conjecturing its general form (as a cubic polynomial in the number of terms) through numerical evidence. Nowadays, a numeric approach can be augmented by a symbolic approach through which such polynomial is generated by software. At the same time, the numeric approach can still be used as the digital tools, augmented by the Online Encyclopedia of Integer Sequences (OEIS®, https://oeis.org/, accessed on September 5, 2024), are capable of converting numeric sequences into their symbolic formulations which can then be modeled numerically by a spreadsheet to verify the conversion through computational triangulation (Abramovich, 2023) that is, by employing more than one digital tool to verify mathematical results.

Using the above-mentioned tools, the paper demonstrates different computational activities with polygonal numbers appropriate for several collegiate mathematics teacher education courses taught either in mathematics or education departments. In particular, those courses may include "Topics in mathematics for elementary teachers", "Contemporary general mathematics", Computer-based mathematics instruction", "Secondary school mathematics instruction", "Topics and research in mathematics education", "History of mathematics" (using the titles of courses taught by the author over the years in master's degree programs in the US states of Georgia, Illinois, and New York). One set of activities discussed below deals with such numbers arranged in equilateral triangles, isosceles triangles, and squares. Depending on the arrangement, a number of computational algorithms for the summation of polygonal numbers – triangular, square, pentagonal and, in the general case, m-gonal, will be developed and discussed.

The paper emphasizes the value of technology-immune/technology-enabled (TITE) mathematical problem solving (Abramovich, 2022) in the modern-day teaching of elementary number theory topics, enabling formal reasoning and digital computation to be integrated in a mutually supportive way across different teacher educational programs. The TITE idea, being used in the present paper as a conceptual framework, is explained in the next section. This framework provides foundation for research and design methods aimed at the inquiry into the teaching of polygonal numbers using variety of educational tools, including digital ones. The paper suggests that the power of modern digital tools allows future teachers of mathematics to appreciate how by using simple algorithms sophisticated computational outcomes can be achieved.

2. Materials and Methods

Two types of materials have been used by the author when working on this paper. The first one is computational; digital tools referred to by mathematics educators in the United States as "mathematical action technologies" (National Council of Teachers of Mathematics, 2014). In the paper, these technologies include comprehensive knowledge engine *Wolfram Alpha* developed by Wolfram Research (www.woframalpha.com; accessed on September 5, 2024) and mathematical software *Maple* (Char et al., 1995). The tools have been used by the author for symbolic computations. One reason of using both tools is mostly pragmatic – *Maple* is convenient for carrying out transformations that involve large algebraic expressions (e.g., difference between two sides of an identity to be proved), something that in the context of *Wolfram Alpha* requires meticulous typing those expressions as they extend the visual size of the program's input box. Another reason is mostly epistemic – the use of more than one digital tool within a single problem provides the result with computational triangulation (Abramovich, 2023) aimed at avoiding both procedural and conceptual errors.

The second type of materials used by the author included teaching and learning mathematics standards used by countries such as Canada (Ontario Ministry of Education, 2020; Western and Northern Canadian Protocol, 2008) Japan (Isoda, 2010; Takahashi, Watanabe, Yoshida, & McDougal, 2006) Singapore (Ministry of Education Singapore, 2020) South Africa (Department of Basic Education, 2018) and the United States (Association of Mathematics Teacher Educators, 2017; Conference Board of the Mathematical Sciences, 2012; National Council of Teachers of Mathematics, 2000, 2014; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The standards uniformly call for integrating mathematical reasoning and digital computations when solving problems. As will be shown in the paper, the context of elementary number theory provides ample opportunities for such integration.

Methods specific for mathematics education used in this paper include computer-based mathematics education, standards-based mathematics, problem solving and problem posing. In particular, those methods are conducive to presenting "teacher candidates with experiences in mathematics relevant to their chosen profession" (Association of Mathematics Teacher Educators, 2017). In the United States, many secondary mathematics teacher preparation programs offer courses "focused on high school mathematics from an advanced standpoint" (Conference Board of the Mathematical Sciences, 2012). Elementary number theory offers ample opportunities to teach advance mathematical ideas with the help of technology. The university where the author has been preparing teacher candidates to teach mathematics is in upstate New York in close proximity to Canada, and many of the author's students are Canadians pursuing their master's degrees in education. As future teachers of mathematics, they learn how to think computationally by "expressing problems in such a way that their solutions can be reached using computational steps and algorithms" (Ontario Ministry of Education, 2020). This diversity of students suggests the importance of aligning mathematics education courses with multiple international perspectives on teaching and learning the subject matter in the digital era.

Problem-solving methods and conceptual methodology used in this paper follow the TITE (technology immune/technology enabled) framework introduced in Abramovich (2022). A TITE problem cannot be automatically solved by software (thus, it is *immune* from the direct use of technology), yet the role of software in solving the problem remains critical (thus, its solution is *enabled* by technology) because, especially "in complex calculations, the effectiveness of learning can be enhanced by using computation tools" (Takahashi et al., 2006). An important aspect of the TITE idea is that when the TI (technology immune) part precedes the TE (technology-enabled) part of problem solving, depending on a problem, there are at least two outcomes available: an error in the TI part would either be neglected or recognized through the action of the TE part. That is why, checking the result of symbolic computations in a special case, the accuracy of which is semi-evident, should be included in a TITE problem solving (see sections 5-7). Likewise, a TI activity may follow a TE one and while the latter would have no error, the former may include erroneous interpretation of the results of the latter. Another important aspect of the TITE idea deals with its duality in a sense that whereas the TE part may inform the TI part, the latter, without conceptual understanding of the former, may lead desired generalization astray. At the same time, the TI part can be used to improve the efficiency of the TE part, which, in turn, supports the advancement of the TI part at the level of generalization. Different TITE problems will be considered in sections 5-7 below.

3. Triangular Numbers

Many mathematics education papers published around the world within the last two decades have been devoted to activities with triangular numbers. Canadian mathematics teacher educators (Simmt, Davis, Gordon, & Towers, 2003) discussed the emergence of triangular numbers in the context of the so-called toothpicks (alternatively, matchsticks (Hershkowitz, Arcavi, & Bruckheimer, 2001) problem. Mathematics educators Pedemonte and Buchbinder (2011) described their work in Italy with upper secondary school students (17-18 years old) towards finding and proving a general rule for the *n*-th triangular number by counting the number of dots in an equilateral triangle with *n* dots on a side representing a triangular number (see the top part of Figure 2). Researchers from Philippines (Berana, Montalbo, & Magpantay, 2015) motivated by work of Asiru (2008) in Nigeria, presented several summation formulas in the context of triangular numbers using the method of mathematical induction for their proof. Bütüner (2016) in the context of upper middle school in Turkey advocated for the use of history of mathematics and concrete materials in the teaching of formulas for the sums of natural, triangular, and square numbers (see formulas (1), (8), (14) below). German mathematics educators (Nührenbörger et al., 2016) described how fourth graders were developing a formula for triangular numbers in the context of substantial learning environments (Wittmann, 2001). A Spanish mathematics educator Plaza (2016) demonstrated visual development of formulas, including the sum of triangular numbers, using the "proof without words" approach. Most recently, Demircioğlu (2023) expressed concern regarding low ability of Turkish future teachers of fourteen-age students in proving the (ready-made rather than mathematically derived) formula for the sum of triangular numbers (see below formula (8) computationally generated by *Wolfram Alpha*) using the method of mathematical induction. Despite their worldwide span and the diversity of contexts, none of the above-mentioned papers used digital technology in the context of triangular and other polygonal numbers. This may serve as a justification for writing the present paper in order to promote the use of digital tools enabling deeper inquiries into the properties of polygonal numbers. In no small part, the paper confirms the viewpoint "mathematics courses that explore elementary school mathematics in depth can be genuinely college-level intellectual experiences, which can be interesting for instructors to teach and for the teachers to take" (Conference Board of the Mathematical Sciences, 2012).

As is well known, triangular numbers (the images of which are shown in Figure 1 and 2) are partial sums of natural numbers (e.g., the *n*-th triangular number t_n is the sum of the first *n* natural numbers where *n* is the rank of the number t_n). Such partial sums are 1, 1 + 2 = 3, 1 + 2 + 3 = 6, 1 + 2 + 3 + 4 = 10, 1 + 2 + 3 + 4 + 5 = 15, and, in general,

$$t_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \qquad (1)$$

There are many interesting properties that triangular numbers t_n satisfy. For example, the difference between two consecutive triangular numbers is the rank of the larger number (the top part of Figure 1) and the sum of two consecutive triangular numbers is a square of the rank of the larger number (bottom part of Figure 1). Using formula (1), the two statements can be immediately verified:

$$t_n - t_{n-1} = \frac{n(n+1)}{2} - \frac{(n-1)n}{2} = n\,, \qquad (2)$$

$$t_n + t_{n-1} = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} = n^2\,. \qquad (3)$$

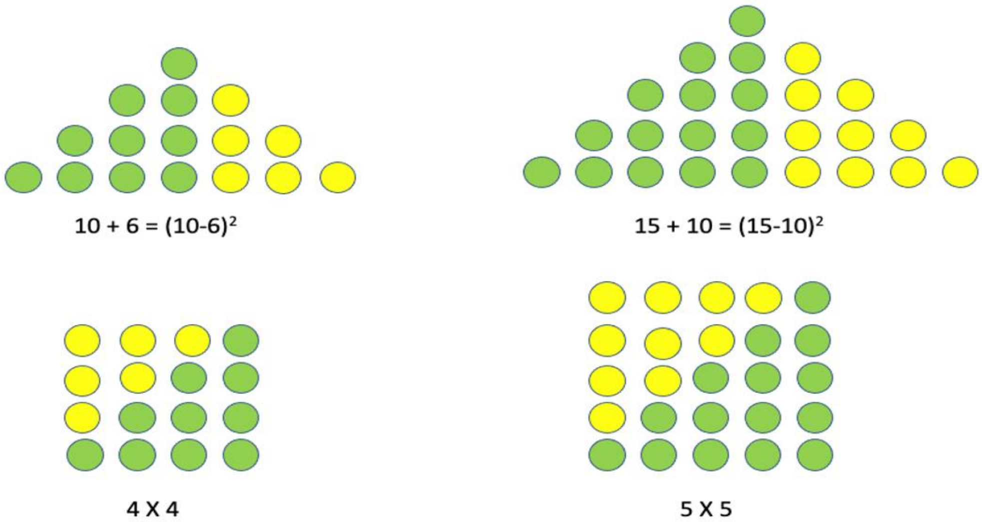

Figure 1. Adding and subtracting two consecutive triangular numbers.

Furthermore, the bottom part of Figure 1 shows how a square can be constructed from circular objects forming two right isosceles triangles the legs of which differ by one circle. In the words of a teacher candidate, the author's student, "*When thinking of square numbers, it is good to have students look at diagrams of triangular numbers first. In looking at triangular numbers, teachers and students can work together to add triangular numbers to form square numbers*". The candidate's focus on working together indicates their appreciation of the constructivist notion of reciprocal learning (Confrey, 1995; Steffe, 1991) which, in turn, motivates the diversity of students' mathematical ideas. In particular, a classroom culture of reciprocal learning can be seen as one of the elements of educational data analysis demonstrating how teachers' leadership skills affect students' engagement and participation in problem solving.

Also, connection between triangular and square numbers is significant from a historical perspective. Leonhard Euler, father of all modern mathematics, in 1778 found an explicit formula for numbers, called square triangular numbers (Dickson, 2005) that are both triangular and square. This is a classic number theory problem known from the time of Diophantus, a 3rd century Greek mathematician. The first four such numbers – 1, 36, 1225, 41616 – when entered into the input box of *Wolfram Alpha* enable the tool to generate their continuation and, in addition, to offer information provided by OEIS®, thereby demonstrating how the use of "mathematical action technologies" (National Council of Teachers of Mathematics, 2014) in the context of teacher education "immensely extends the possibility of behavior by making the results of the work by geniuses available to everyone" (Vygotsky, 1930). Furthermore, connection between triangular and square numbers represented through relation (3) was used by Élie de Joncourt, an 18th century Dutch minister of church and mathematics teacher, to compute squares and square roots (Roegel, 2013). Including mathematical perspectives in a course for teacher candidates allows them to appreciate "mathematics as a living and evolving subject" (Conference Board of the Mathematical Sciences, 2012). As mentioned by another teacher candidate, "*I think understanding the history of mathematics might make understanding why we do it and why it is important easier. When you can put a starting point or reason behind something, I think understanding naturally follows more easily*". This comment by a future teacher of mathematics rhymes with professional opinions about educative significance of the subject matter's history which "keeps students active ... by facilitating various uses of historical content" (Bütüner, 2016) and "makes mathematics more interesting, more understandable, and more approachable" (Fried, 2001). Moreover, using technology makes it easier to talk about "a starting point" that over the centuries evolved into today's choice of mathematics education topics the instrumental value of which is in promoting discovery (McEwan & Bull, 1991).

Any triangular number is a sum of a multiple of three and another triangular number. Indeed, as shown in Figure 2 (the bottom part), $10 = 3 \times 3 + 1, 15 = 3 \times 4 + 3, 21 = 3 \times 5 + 6$. Such representations may not be unique for a triangular number. Already, $15 = 3 \times 3 + 6 = 3 \times (6 - 3) + 6$ and $21 = 3 \times 2 + 15.$ Teacher candidates, especially at the elementary level, do not have much experience with questions allowing for more than one correct answer. At the same time, they are expected to "tailor instruction in ways that build on what students understand and consistently encourage students to stretch their mathematical thinking" (Association of Mathematics Teacher Educators, 2017). However, among such multiple representations there is always a representation in which the number repeated three times is one smaller than the rank of the triangular number being represented; alternatively, the difference between two triangular numbers located between the represented one and the addend in the right-hand side of the relation in question. For example, in the representation of the 5th triangular number 15 the number 4 is the rank of the triangular number 10 and, according to relation (2), the equality $4 = 10 - 6$ holds true; alternatively, 6 and 10 are triangular numbers located between 3 and 15. That is, one can write $15 = 3 \times (10 - 6) + 3$. Likewise, in the representation $21 = 3 \times 5 + 6$ the number $5 = 6 - 1 = 15 - 10$. That is, $21 = 3 \times (15 - 10) + 6$. One can generalize from the diagrams of Figure 2 to have the relation

$$t_n = 3 \times (t_{n-1} - t_{n-2}) + t_{n-3}. \qquad (4)$$

Proof of relation (4) is based on formula (1) and relation (2). Using (1) and (2) yields

$\frac{n(n+1)}{2} = 3 \times (n-1) + \frac{(n-3)(n-2)}{2}$ whence $n^2 + n = 6n - 6 + n^2 - 5n + 6$. Because the last equality holds true for all n, so does relation (4).

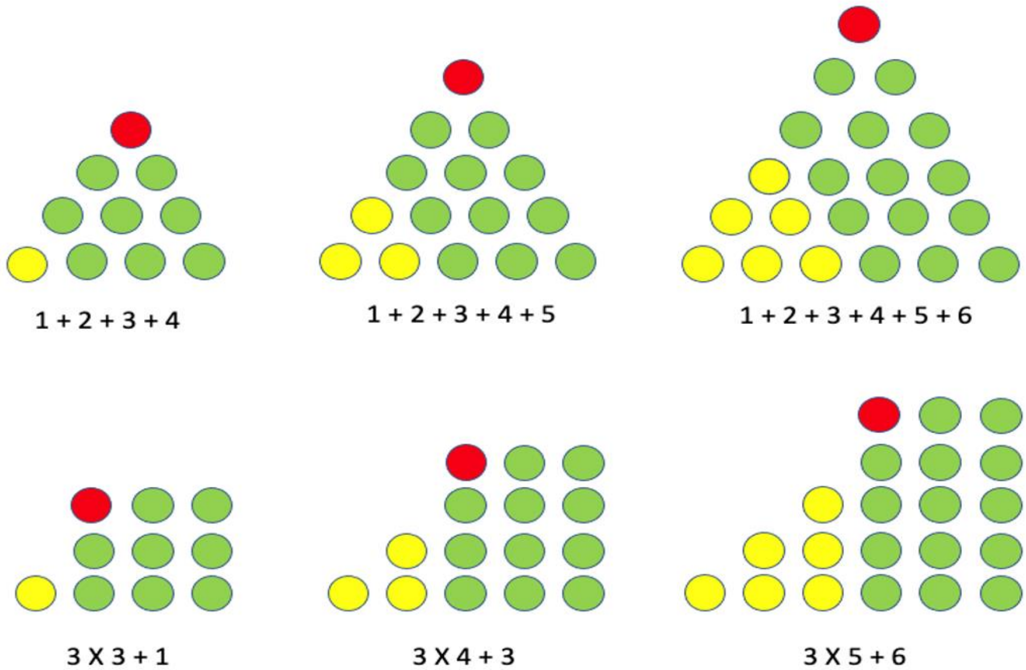

Figure 2. Triangular numbers as sums of a multiple of three and a triangular number.

Remark 1. One might think that the divisibility by three observation is due to the fact that triangular numbers are represented by triangles (top part of Figure 2) and that a triangular number is either a multiple of three or one greater than a multiple of three (bottom part of Figure 2). However, in the case of square numbers 1, 4, 9, 16, 25, 36, 49 ..., one can recognize similar relationships (including more than one) such as $16 = 3 \times 5 + 1 = 3 \times (9 - 4) + 1$, $25 = 3 \times 7 + 4 = 3 \times (16 - 9) + 4$, $36 = 3 \times 9 + 9 = 3 \times (25 - 16) + 9$, $49 = 3 \times 11 + 16 = 3 \times (36 - 25) + 16$, which show that a square number is a sum of an odd multiple of three plus another square number (Figure 3 shows a possible representation of 25 as 7 + 7 + 7 + 4). Although $16 = 3 \times 4 + 4$ and $25 = 3 \times 8 + 1$, neither 4 nor 8 is an odd number – the only possible difference between two consecutive square numbers and in that sense the last two representations of 16 and 25 are different from those with differences 5 and 7. One can check to see that the same type of relationships can be developed for pentagonal numbers 1, 5, 12, 22, 35, 51, ... For example, $51 = 3 \times 13 + 12 = 3 \times (35 - 22) + 12$. In general, among four consecutive polygonal numbers of side m the following relation holds true

$$P(m,n) = 3 \times \big(P(m,n-1) - P(m,n-2)\big) + P(m,n-3). \qquad (5)$$

Here $P(m,n) = \frac{n^2(m-2) - n(m-4)}{2}$ – an m-gonal number of rank n. The proof of relation (5) can be outsourced to *Maple* (Figure 4). To clarify the use of *Maple*, note that everything that needs to be typed in the program here and elsewhere when proving algebraic identities is included after the > sign with calculations followed; the percentage symbol in the *Maple* language means "the latter"; simplification yields zero; in particular, the first typing introduces the expression for $P(m,n)$ and the second typing defines the difference between the left- and the right-hand sides of (5) to be computed symbolically. As will be shown in the next section, the factor three appearing in (4) and (5) is due to the length of a string of the polygonal numbers considered. One can say that the factor three represents the invariance as a property of any four consecutive m-gonal numbers that remains unchanged when the value of m changes. In the words of Singaporean mathematics educators, teaching that focuses on big ideas "brings one closer to appreciating the nature of mathematics" (Ministry of Education Singapore, 2020). Recognition of invariance is preceded by recognition of patterns, something that is commonly described as the essence of mathematics (Delvin, 1994; National Council of Teachers of Mathematics, 2000; Resnik, 1999; Western and Northern Canadian Protocol, 2008). As one teacher candidate, another student by the author, exclaimed, "*I have been shocked and surprised by some of the patterns that are sitting right in front of me that I was not aware of! It is pretty interesting how some brains are different than others and pick up on patterns so easily!*". As will be shown below through patterns' recognition, the change of the (even) length of a string of *m*-gonal numbers yields a new (odd) factor.

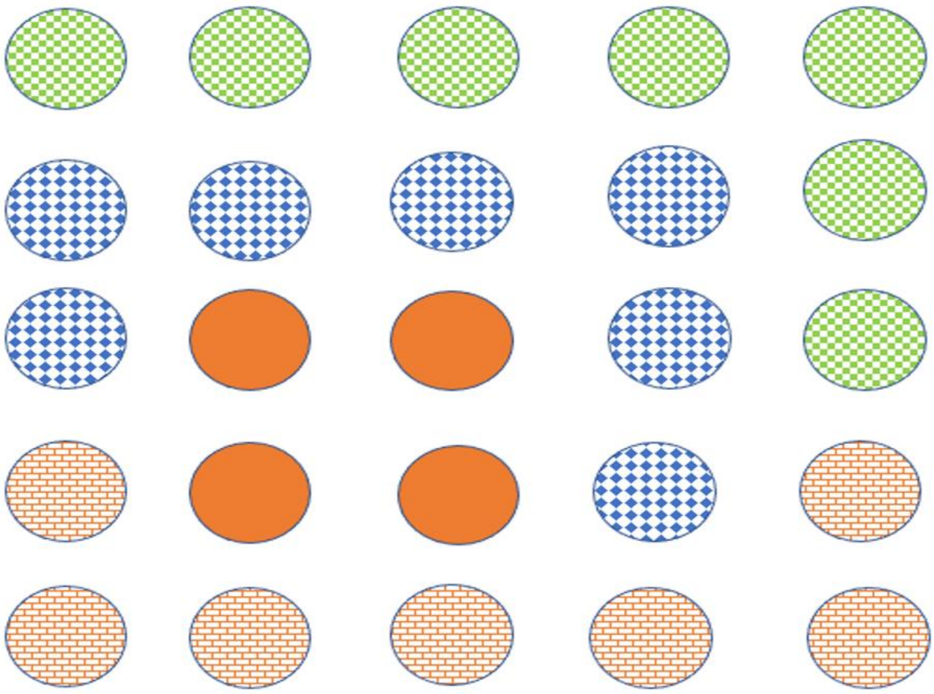

Figure 3. Visual demonstration of the equality $25 = 3 \times 7 + 4$.

> $P(m, n) := \dfrac{n^2 \cdot (m-2) - n \cdot (m-4)}{2}$

$$P := (m, n) \mapsto \frac{n^2 \cdot (m-2)}{2} - \frac{n \cdot (m-4)}{2}$$

> $P(m, n) - 3 \cdot (P(m, n-1) - P(m, n-2)) - P(m, n-3)$

$$\frac{n^2 (m-2)}{2} - \frac{n (m-4)}{2} - \frac{3 (n-1)^2 (m-2)}{2} + \frac{3 (n-1) (m-4)}{2} + \frac{3 (n-2)^2 (m-2)}{2} - \frac{3 (n-2) (m-4)}{2} - \frac{(n-3)^2 (m-2)}{2} + \frac{(n-3) (m-4)}{2}$$

> *simplify*(%)

$$0$$

Figure 4. Using *maple* in proving relation (5).

4. Generalizing from Formula (5)

If one considers the first six triangular numbers 1, 3, 6, 10, 15, 21, then, as shown at the left-hand side of Figure 5, the number 21 (= 1 + 2 + ... + 6) is one greater than a multiple of five, i.e., $21 = 5 \times 4 + 1 = 5 \times (10 - 6) + 1$. Likewise, if one considers the first eight triangular numbers 1, 3, 6, 10, 15, 21, 28, 36, then, as shown at the right-hand side of Figure 5, the number 36 (= 1 + 2 + ... + 8) is one greater than a multiple of seven, i.e., $36 = 7 \times 5 + 1 = 7 \times (15 - 10) + 1$. One can check to see that similar relationships hold true in the case of other strings of six and eight consecutive triangular numbers, respectively. Generalizing from such numeric equalities, yields two symbolic relations $t_n = 5(t_{n-2} - t_{n-3}) + t_{n-5}$ and $t_n = 7(t_{n-3} - t_{n-4}) + t_{n-7}$ that hold true among four numbers selected out of, respectively, six and eight consecutive triangular numbers (see Figure 5, left and right, for $n = 6$ and $n = 8$, respectively). Furthermore, the following generalization can be inductively formulated for four numbers (the first, the last, and two in the middle) selected out of $2k + 2$ consecutive *m*-gonal numbers:

$$P(m, n) = (2k + 1)\big(P(m, n-k) - P(m, n-k-1)\big) + P(m, n-2k-1). \qquad (6)$$

Once again, the proof of relation (6) can be outsourced to *Maple* (Figure 6; as was mentioned above, the percentage symbol in the *Maple* language means "the latter"; simplification yields zero). One can say that the two consecutive integers $2k + 1$ and $2k + 2$ represent the invariance as the property, expressed by relation (6), of any string of the latter length of consecutive m-gonal numbers that remains unchanged as m changes and the case $k = 1$ expressed by relation (5) is a special case of this invariance.

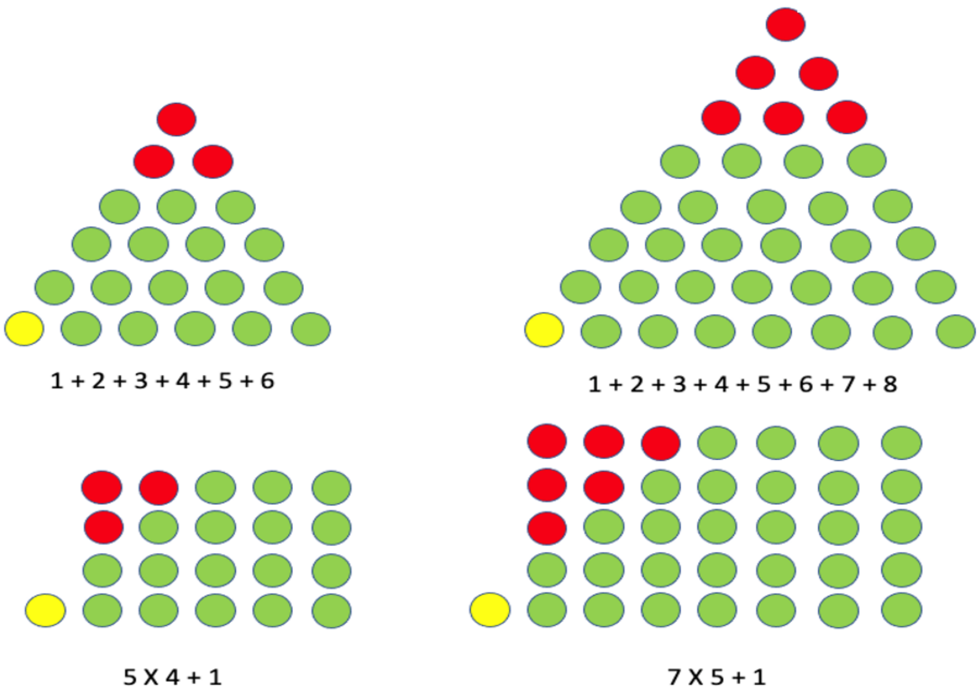

Figure 5. Triangular numbers as sums of multiples of five and seven plus one.

> $P(m,n) := \frac{n^2\cdot(m-2) - n\cdot(m-4)}{2}$

$$P := (m,n) \mapsto \frac{n^2\cdot(m-2)}{2} - \frac{n\cdot(m-4)}{2}$$

> $P(m,n) - (2k+1)\cdot(P(m,n-k) - P(m,n-k-1)) - P(m,n-2k-1)$

$$\frac{n^2(m-2)}{2} - \frac{n(m-4)}{2} - (2k+1)\left(\frac{(n-k)^2(m-2)}{2} - \frac{(n-k)(m-4)}{2} - \frac{(n-k-1)^2(m-2)}{2} + \frac{(n-k-1)(m-4)}{2}\right) - \frac{(n-2k-1)^2(m-2)}{2} + \frac{(n-2k-1)(m-4)}{2}$$

> $simplify(\%)$

$$0$$

Figure 6. Using *maple* to prove formula (6).

5. Constructing Triangles using Triangular Numbers as a TITE Problem

In this section, numeric and symbolic activities with triangular numbers will be integrated with their arrangement in triangular shapes using the TITE framework (explained in detail in section 2). One such integration is when visual enhanced by numeric enables transition to symbolic number theory explorations at the level of generalization. However, due to the duality of the TITE framework, when the TE part not only informs the TI part, but without conceptual understanding of what technology is supposed to deliver, the TI part might go astray at the level of generalization, the following (teacher education) classroom episode is worth noting.

A teacher candidate, when dealing with the following four numbers 2, 3, 5, 8 (after finding that among the binary numbers in which 1's are always separated by at least one 0, there are exactly 2, 3, 5 and 8 of one-, two-, three-, and four-digit binary strings, respectively), and being asked to inductively continue the sequence by recognizing a pattern, recognized (instead of Fibonacci recursion) that each term beginning from the second augments a triangular number by two ($3 = 1 + 2$, $5 = 3 + 2$, $8 = 6 + 2$), continued the sequence with the numbers 12 ($= 10 + 2$), 17 ($= 15 + 2$), 23 ($= 21 + 2$), and finally concluded that the sequence has the form $\frac{n(n-1)}{2} + 2 = \frac{n-n+4}{2}, n = 1, 2, 3, \ldots$. At the same time, after being introduced to a special ready-made spreadsheet programmed to generate binary numbers (not discussed here for its complexity), the candidate found out that the number of five-digit binary strings in which 1's are always separated by at least one 0 is 13 and not 12. It is interesting to note that when the four numbers 2, 3, 5, 8 are entered into the input box of *Wolfram Alpha*, the tool offers continuation in the form $a(n) = \frac{n-n+4}{2}$. This episode suggests that transition from numeric to symbolic (or from visual to symbolic) would be more accurate when one possess some initial understanding of how the symbolic should look like (Arnheim, 1969). On the other hand, accepting the above continuation (of what might be "obvious" for some as four consecutive Fibonacci numbers) supports an educational recommendation that when dealing with students whose thinking is different from what is expected, one should have intellectual courage to acknowledge the unexpected "as strength and resources upon which to build" (Association of Mathematics Teacher Educators, 2017).

TITE mathematical problem solving may include the use of multiple software tools in support of a single task to enable rigor in problem solving through computational triangulation (Abramovich, 2022). In many cases, using different tools—numeric, symbolic, graphic—not only provides rigor in solving a problem but it also motivates one's deep thinking about computationally obtained results that can lead to formulating new problems. Furthermore, the diversity of digital tools fosters computational thinking (Wing, 2006) which, among other "transferable skills ... [is] in high demand in today's globally connected world, with its unprecedented advancement in technology" (Ontario Ministry of Education, 2020). At the same time, because, in the words of mathematics educators in South Africa (Department of Basic Education, 2018) "mathematics teachers, and not ICT [Information and Communication Technology] tools, are the key to quality education", students' growth in computational thinking and awareness of technological diversity requires a "more knowledgeable other" (Vygotsky, 1978) in a mathematics classroom.

As the first example of a TITE number theory problem, consider an equilateral triangle (the top-right part of Figure 2) which represents the development of triangular numbers as partial sums of consecutive natural numbers. Here, each natural number is represented by a row of circles so that the partial sums are 1 = 1, 3 = 1 + 2, 6 = 1 + 2 + 3, ..., 21 = 1 + 2 + ... + 6. The representation of 21 as the 6th partial sum can be interpreted under the assumption that each circle has numerical value equal to one. But in the case of alternative numeric assignments for the circles, one can consider partial sums of the numbers as being assigned to the circles. One can assign to each circle a triangular number so that the entire triangle is filled with consecutive triangular numbers. Figure 7 shows such assignment for an equilateral triangle comprised of six rows. Let $S_{n,1}^{3}$ be the sum of the numbers in the first *n* rows of the (extended) triangle of Figure 7. In the notation $S_{n,1}^{3}$, the top index 3 points at triangular numbers, the bottom indices *n* and 1 point, respectively, at the number of rows in the sum and the difference between the number of circles in two neighboring rows. Finding a formula for $S_{n,1}^{3}$ and computing its symbolic expression represent a TITE problem.

To solve the problem under TITE umbrella, one begins with constructing a formula (a TI step) for computing $S_{n,1}^{3}$ (a TE step). In Figure 7 there are six rows with the far-right number being the triangular number the rank of which is also a triangular number with the rank being the column's number. For example, in Figure 7 the far-right number in row 2 is $6 = t_3 = t_{t_2}$, the far-right number in row 3 is $21 = t_6 = t_{t_3}$, the far-right number in row 4 is $55 = t_{10} = t_{t_4}$, and so on. If the far-right number in row *n* (having *n* circles) is t_{t_n}, then row $(n + 1)$ has the next triangular number t_{t_n+1} at the far-left and the number $t_{t_{n+1}}$ at the far-right. Indeed, if the far-left number in row $(n + 1)$ with $(n + 1)$ circles is t_{t_n+1}, then the next number in this row is t_{t_n+2}, followed by the number t_{t_n+3}, ..., so that the far-right number in this row is $t_{t_n+(n+1)}$. Due to formula (2), $t_{n+1} - t_n = n + 1$ and, therefore, $t_{t_n+(n+1)} = t_{t_{n+1}}$. So, the far-right number in the *n*-th row is t_{t_n}. That is,

$$S_{n,1}^{3} = 1 + 3 + 6 + \cdots + t_{t_n}. \quad (7)$$

The right-hand side of relation (7) represents the sum of the first t_n triangular numbers. But first, one has to find the sum of the first *n* triangular numbers. Nowadays, such sum can be found using *Wolfram Alpha* (Figure 8) and formula (1) to get

$$1 + 3 + 6 + \cdots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}. \quad (8)$$

In order to complete summation in (7), one either can use formula (8) in which *n* has to be replaced by $t_n = \frac{n(n+1)}{2}$ (TI step, envisioning the sum to be a polynomial of degree six) to get $S_{n,1}^{3} = \frac{n(n+1)(n^2+n+2)(n^2+n+4)}{48}$, or do it computationally (TE step) by entering into the input box of *Wolfram Alpha* the command "sum i(i+1)/2 for i=1 to n(n+1)/2". (Note that every screen shot of the program pictured in this paper includes the expression of summation, like i(i+1)/2, as well as its boundaries, like 1 and n(n+1)/2, aiding similar computations). As shown in Figure 9, the TE step quickly processes the corresponding TI step and yields the relation

$$S_{n,1}^{3} = \sum_{i=1}^{\frac{n(n+1)}{2}} \frac{i(i+1)}{2} = \frac{n(n+1)(n^2+n+2)(n^2+n+4)}{48}. \quad (9)$$

As a kind of verification of the accuracy of the above TITE problem solving, when a TE step simply executed the preceding TI step, one can check to see that formula (9) generates 10 when *n* = 2. Indeed, $S_{2,1}^{3} = \frac{2\times3\times8\times10}{48} = 10$ – the sum of numbers in the first two rows of Figure 7. Other TITE problems will be suggested in sections 6 and 7 below.

Remark 2. As noted by mathematics educators in Japan, “algebraic expressions ... may be modified into a form that is more easily interpreted” (Isoda, 2010). With this in mind, comparing formulas (8) and (9), one can conclude that whereas *the sum of the first n triangular numbers is 1/6 of the product of three consecutive integers starting from n, the sum of the first t_n triangular numbers is 1/6 of the product of three consecutive integers starting from* $t_n = \frac{n(n+1)}{2}$. Indeed, $\frac{n(n+1)(n^2+n+2)(n^2+n+4)}{48} = \frac{\frac{n(n+1)}{2}\left(\frac{n(n+1)}{2}+1\right)\left(\frac{n(n+1)}{2}+2\right)}{6}$. For example, when $n = 4$ we have $t_4 = 10$ and the sum of the first ten triangular numbers $1 + 3 + 6 + \cdots + 55 = \frac{10\times11\times12}{6} = 220$.

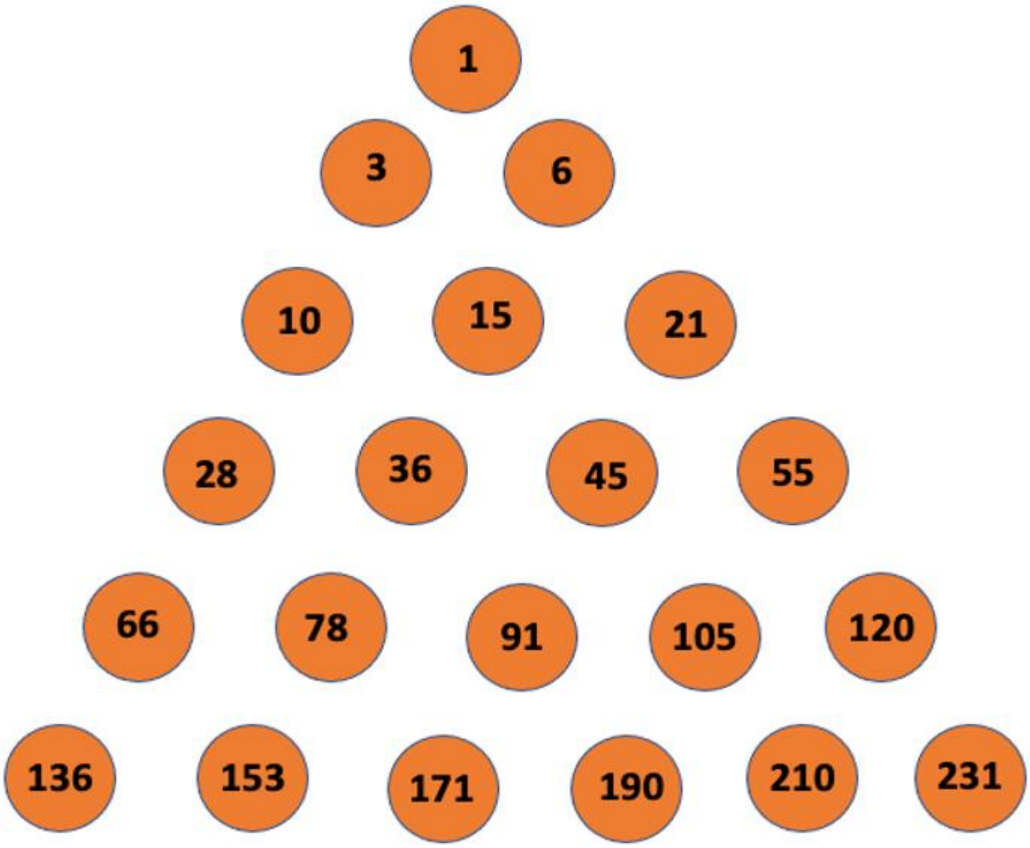

Figure 7. An equilateral triangle filled with consecutive triangular numbers.

Sum

$$\sum_{i=1}^{n} \frac{1}{2} i(i+1) = \frac{1}{6} n(n+1)(n+2)$$

Figure 8. Calculating the sum of the first n triangular numbers.

Sum

$$\sum_{i=1}^{\frac{1}{2}n(n+1)} \frac{1}{2} i(i+1) = \frac{1}{48} n(n+1)\left(n^2+n+2\right)\left(n^2+n+4\right)$$

Figure 9. Calculating the sum of the first t_n triangular numbers.

6. Constructing Triangles using Square Numbers as a TITE Problem

Just as in the case of triangular numbers, a problem can be posed to investigate the arrangement of square numbers in triangles having rows with lengths forming an arithmetic progression. Let $S_{n,k}^4$, $(k = 1,2,3)$, be the sum of the numbers in the first *n* rows of the (extended) triangles of Figure 10 and 11, respectively. Finding symbolic expressions for the three sums is a TITE problem. As shown in the diagram of Figure 10, a mathematical analysis of which (and the diagrams that follow) is a TI part of the problem, when the difference between two consecutive lengths is equal to one, the far-right number in the *n*-th row is the square of the triangular number of rank *n*. Therefore, using *Wolfram Alpha* as a TI part of the problem, one can find that the sum of numbers in *n* rows of the extended diagram of Figure 10 is equal to

$$S_{n,1}^4 = \frac{(n^2+n)(n^2+n+1)(n^2+n+2)}{24}. \quad (10)$$

As shown in the diagram of Figure 11 (top), when the difference between two consecutive lengths is equal to two, the far-right number in the *n*-th row is the square of n^2. Therefore, using *Wolfram Alpha* as a TI part of the problem, one can find that the sum of numbers in *n* rows of the extended diagram of Figure 11 (top) is equal to

$$S_{n,2}^4 = \frac{n^2(n^2+1)(2n^2+1)}{6}. \quad (11)$$

As shown in the diagram of Figure 11 (bottom), when the difference between two consecutive lengths is equal to three, the far-right number in the *n*-th row is the square of the pentagonal number of rank *n*. Therefore, using *Wolfram Alpha* as a TI part of the problem, one can find that the sum of numbers in *n* rows of the extended diagram of Figure 11 (bottom) is equal to

$$S_{n,3}^4 = \frac{(3n^2-n)(3n^2-n+1)(3n^2-n+2)}{24}. \quad (12)$$

Similarly, using *Wolfram Alpha*, one can compute

$$S_{n,4}^4 = \frac{(2n^2-n)(2n^2-n+1)(4n^2-2n+1)}{6}. \quad (13)$$

Remark 3. The right-hand sides of relations (10) – (13) can be modified to be written in the form of the well-known formula

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \quad (14)$$

Indeed, in the case of relation (10), setting $k = \frac{n^2+n}{2}$ – the triangular number of rank *n*, we have $\frac{(n^2+n)(n^2+n+1)(n^2+n+2)}{24} = \frac{k(k+1)(2k+1)}{6}$. In other words, when the number of addends in the sum of the first square numbers is the *n*-th triangular number t_n, this sum is equal to 1/6 of the three-factor product $t_n \times (t_n + 1) \times (2t_n + 1)$. In particular, when $n = 2$ we have $t_2 = 3$ and $1 + 4 + 9 = \frac{3\times4\times7}{6} = 14.$ In the case of relation (11), setting $k = n^2$ – the square number of rank *n*, we have $\frac{n^2(n^2+1)(2n^2+1)}{6} = \frac{k(k+1)(2k+1)}{6}$. In other words, when the number of addends in the sum of the first square numbers is the *n*-th square number s_n, this sum is equal to 1/6 of the three-factor product $s_n \times (s_n + 1) \times (2s_n + 1)$. In particular, when $n = 2$ we have $s_2 = 4$ and $1 + 4 + 9 + 16 = \frac{4\times5\times9}{6} = 30.$ In the case of relation (12), setting $k = \frac{3n^2-n}{2}$ – the pentagonal number of rank *n*, we have $\frac{(3n^2-n)(3n^2-n+1)(3n^2-n+2)}{24} = \frac{k(k+1)(2k+1)}{6}$. In other words, when the number of addends in the sum of the first square numbers is the *n*-th pentagonal number p_n, this sum is equal to 1/6 of the three-factor product $p_n \times (p_n + 1) \times (2p_n + 1)$. In particular, when $n = 2$ we have $p_2 = 5$ and $1 + 4 + 9 + 16 + 25 = \frac{5\times6\times11}{6} = 55.$ In the case of relation (13), setting $k = 2n^2 - n$ – the hexagonal number of rank *n*, we have $\frac{(2n^2-n)(2n^2-n+1)(4n^2-2n+1)}{6} = \frac{k(k+1)(2k+1)}{6}$ (Figure 12). In other words, when the number of addends in the sum of the first square numbers is the *n*-th hexagonal number h_n, this sum is equal to 1/6 of the three-factor product $h_n \times (h_n + 1) \times (2h_n + 1)$. In particular, when $n = 2$ we have $h_2 = 6$ and $1 + 4 + 9 + 16 + 25 + 36 = \frac{6\times7\times13}{6} = 91.$ In general, setting $k = \frac{n^2(m-2)-n(m-4)}{2}$ – the *m*-gonal number of rank *n*, the use of *Wolfram Alpha* (Figure 13) in finding $\sum_{i=1}^{\frac{n^2(m-2)-n(m-4)}{2}} i^2$ yields the relation $\frac{(n^2(m-2)-n(m-4))(n^2(m-2)-n(m-4)+1)(n^2(m-2)-n(m-4)+2)}{24} = \frac{k(k+1)(2k+1)}{6}$. In other words, when the number of addends in the sum of the first square numbers is the *n*-th *m*-gonal number $P(m,n)$, this sum is equal to 1/6 of the three-factor product $P(m,n) \times (P(m,n) + 1) \times (2P(m,n) + 1)$.

Remark 4. Representing the right-hand sides of relations (12) and (13) in the form $\frac{k(k+1)(2k+1)}{6}$ may be seen as the first step towards proving that the corresponding products are multiples of six. To complete the proof, one has to assume that $k(k + 1)(2k + 1)$ is a multiple of 6 and then make an inductive transfer from *k* to $k + 1$ by noting that $(k + 1)(k + 2)(2k + 3) - k(k + 1)(2k + 1) = 6(k + 1)^2$ thereby implying that transition from k to $k + 1$ preserves divisibility by six. Note that the content of the last two remarks shows how a TI part may follow a TE part within a TITE problem.

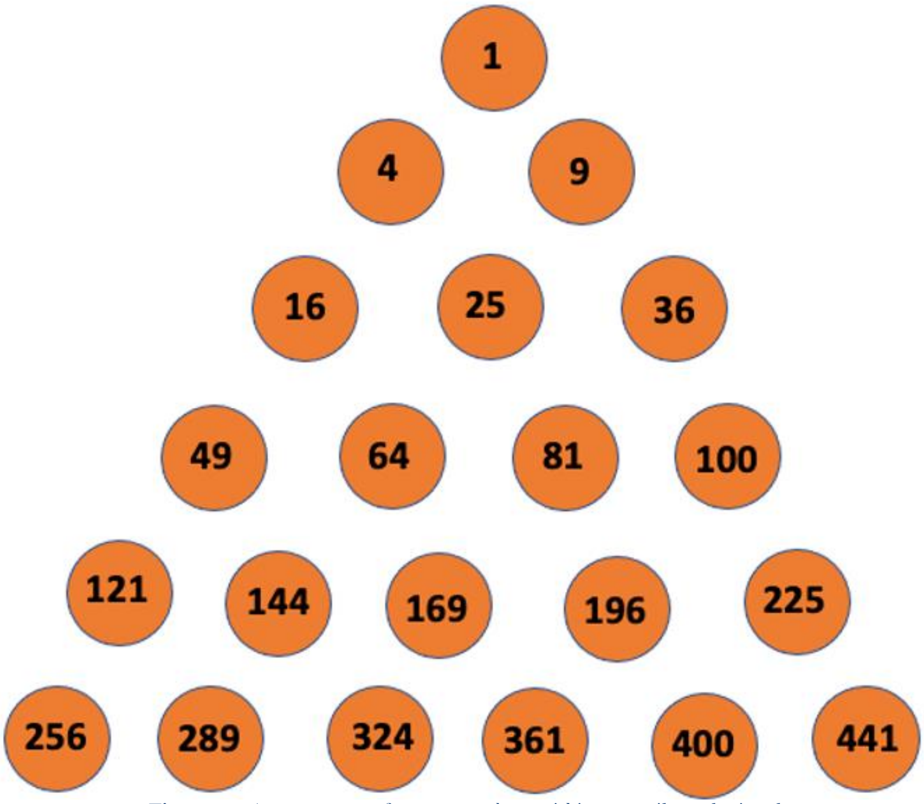

Figure 10. Arrangement of square numbers within an equilateral triangle.

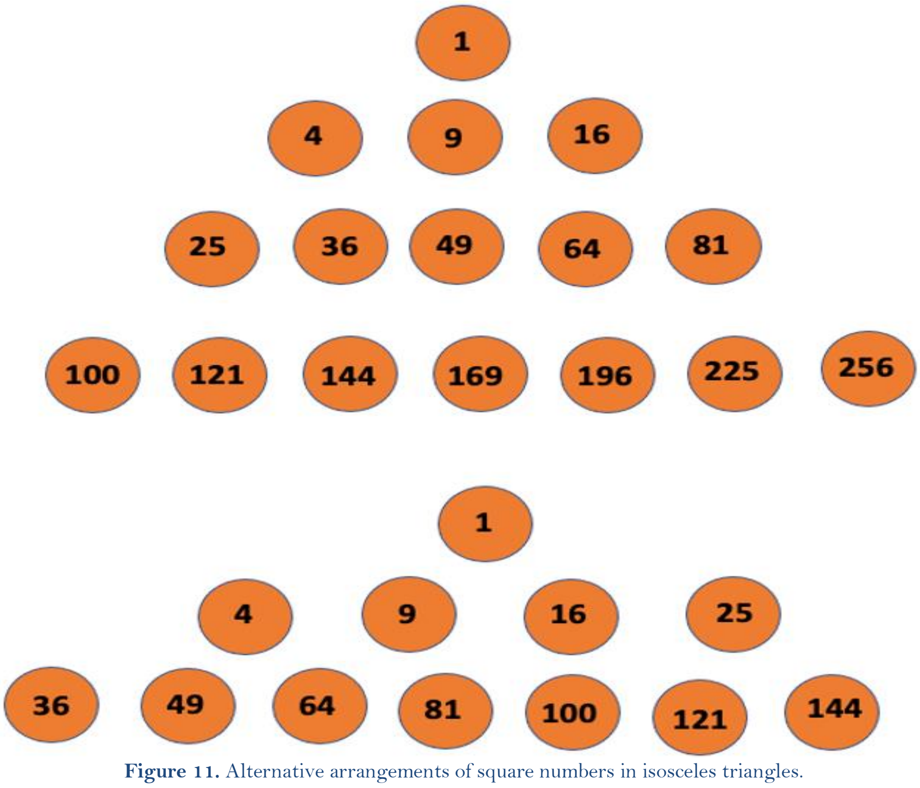

Figure 11. Alternative arrangements of square numbers in isosceles triangles.

Sum

$$\sum_{i=1}^{n(2n-1)} i^2 = \frac{1}{6} n(2n-1)(2n^2-n+1)(4n^2-2n+1)$$

Figure 12. Calculating the sum of the first h_n square numbers.

Sum

$$\sum_{i=1}^{\frac{1}{2}((m-2)n^2-(m-4)n)} i^2 = \frac{1}{24} n(m(n-1)-2n+4)((m-2)n^2-(m-4)n+1)((m-2)n^2-(m-4)n+2)$$

Figure 13. Calculating the sum of the first $P(m,n)$ square numbers.

7. Suggestions for Further Explorations

Other geometric arrangements of polygonal numbers can be explored. Suggested explorations can be used as independent capstone projects in technology-enhanced problem-solving courses. Such projects allow secondary teacher candidates to "take up classical ideas that are not normally included in a mathematics major" (Conference Board of the Mathematical Sciences, 2012) and explore them through the TITE lens to see "how reasoning and proof occur through the high school mathematics outside of their traditional home in axiomatic Euclidean geometry" (ibid: 59). For example, consecutive triangular numbers can be arranged in squares forming a staircase the stairs of which increase in size by one, two, three, four, etc. rows of circles. Consider the case of the sequence of squares $(2n-1)\times(2n-1), n = 1, 2, 3, \ldots.$, in which consecutive odd numbers are the dimensions of the squares (Figure 14). The ranks of triangular numbers assigned to the first and the last circles in this sequence of squares are (1, 1), (2, 10), (11, 35), (36, 84), (85, 165), ... In general, the first and the second elements of those pairs are described by the sequences (found through *Wolfram Alpha*) $a_n = \frac{n(4n^2-12n+11)}{3}$ and $b_n = \frac{n(4n^2-1)}{3}$, respectively. Indeed, $a_3 = 11, b_3 = 35.$ That is, the square of rank *n* in the sequence $(2n-1)\times(2n-1)$ would begin and end with the triangular numbers $t_{a_n} = \frac{a_n(a_n+1)}{2}$ and $t_{b_n} = \frac{b_n(b_n+1)}{2}$. A TITE problem is to prove the relation

$$t_{b_n+1} = t_{a_{n+1}}. \qquad (15)$$

Computations can be outsourced to *Maple* (Figure 15; as was mentioned above, the percentage symbol in the *Maple* language means "the latter"; simplification yields zero).

Likewise, the squares can be filled with squares and pentagonal numbers, so that in the case of the sequence of squares $(2n-1)\times(2n-1)$ the following relations can be proved:

$$s_{b_n+1} = s_{a_{n+1}} \qquad (16)$$

and

$$p_{b_n+1} = p_{a_{n+1}}\, , \qquad (17)$$

Where $s_{a_n} = (a_n)^2$ and $p_{a_n} = \frac{n(3n-1)}{2}$.

A TITE characterization of problems associated with relations (16) and (17) is in using mathematical reasoning in their formulation and technology in their proving. In fact, both TI and TE parts of the problems go hand in hand because an error in mathematical reasoning would be recognized by a digital tool through technological rejection of an identity as an erroneous mathematical formulation. This type of TITE problem on proving an identity is different from the problem of section 5 on finding a certain sum of triangular numbers which resulted in the development of relation (7) the correctness of which hinged on the verification of a special case with reference to Figure 7.

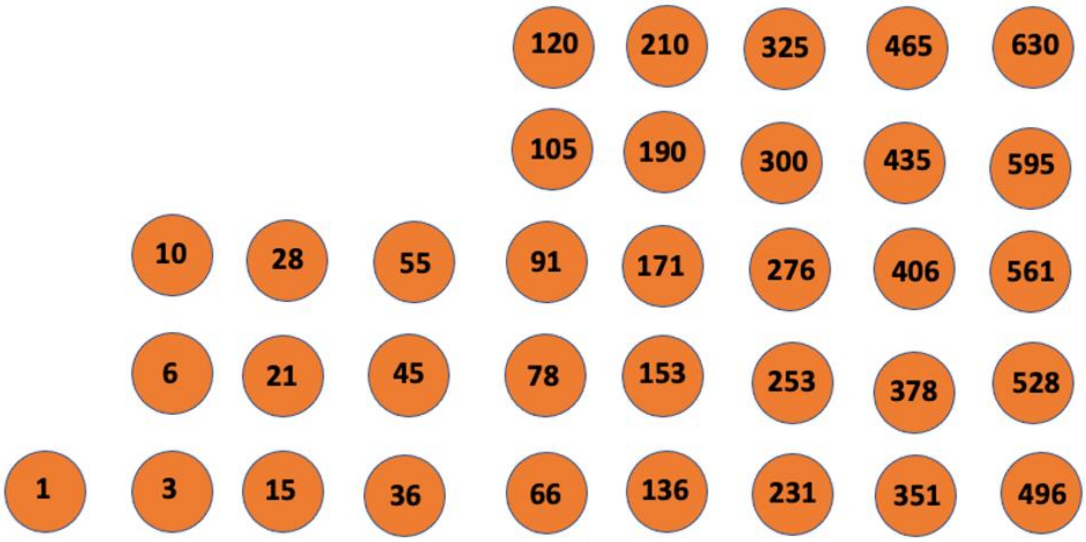

Figure 14. Triangular numbers arranged in squares forming a staircase.

$$> a(n) := \frac{n\cdot(4\cdot n^2 - 12\cdot n + 11)}{3}$$

$$a := n \mapsto \frac{n\cdot(4\cdot n^2 - 12\cdot n + 11)}{3}$$

$$> b(n) := \frac{n\cdot(4\cdot n^2 - 1)}{3}$$

$$b := n \mapsto \frac{n\cdot(4\cdot n^2 - 1)}{3}$$

$$> \frac{(b(n)+1)\cdot(b(n)+2)}{2} - \frac{a(n+1)\cdot(a(n+1)+1)}{2}$$

$$\frac{\left(\frac{n\,(4\,n^2-1)}{3}+1\right)\left(\frac{n\,(4\,n^2-1)}{3}+2\right)}{2} - \frac{(n+1)\left(4\,(n+1)^2-12\,n-1\right)\left(\frac{(n+1)\left(4\,(n+1)^2-12\,n-1\right)}{3}+1\right)}{6}$$

$$> simplify(\%)$$

$$0$$

Figure 15. Using *Maple* to prove identity (15).

8. Conclusion

The paper dealt with technology-supported elementary number theory activities in a mathematics teacher education context. Reviewing recently published mathematics education research from Canada, Germany, Italy, Nigeria, Philippines, Spain and Turkey revealed no use of digital tools in teaching triangular (and other polygonal) numbers either at the college or pre-college levels. This absence of computers in the educational context of triangular numbers was a motivation for writhing the paper. Different geometric arrangements of polygonal numbers were explored. The arrangements included equilateral and isosceles triangles the shapes of which were defined by arithmetical sequences. Computational summation of numbers included in those arrangements was supported by *Maple* and *Wolfram Alpha*. It is the complexity of such arrangements forming integrated patterns defined by number sequences organized in geometric shapes that required symbolic computations for understanding mathematical behavior of the patterns. Ideas for further explorations involving arrangements of polygonal numbers into squares forming a staircase were briefly discussed and suggested for independent capstone projects.

It was shown how number theory and geometry as different branches of mathematics can be united under an umbrella enriched by historical perspectives (Jankvist, 2009). An emphasis on multiple representations of ideas associated with polygonal numbers demonstrated how students, by "recognizing, describing, and working with numerical and non-numerical patterns ... develop fluency in moving from one representation to another" (Western and Northern Canadian Protocol, 2008). Comments of teacher candidates, the author's students, related to pattern recognition and the value of history in mathematics teaching were shared as appropriate. Those comments can be interpreted as qualitative evidence (collected by the author over the years through different means) of the value of digital tools in teaching topics from elementary number theory to teacher candidates.

Polygonal numbers provide context that integrates symbolic artistry and visual understanding in mathematical problem solving. The importance of such integration was recognized by Jean-Jacques Rousseau, a Genevan philosopher and writer of the 18th century, in his autobiographical book Confessions (https://en.wikipedia.org/wiki/Confessions_(Rousseau)), "The first time I found by calculation that the square of a binomial consisted of the squares of its two parts plus twice the product of the two, I refused to believe it until I has drawn the figure" (cited in Arnheim (1969)). Whereas this approach can be traced even deeper in the history of civilization – as Aristotle in the 4th century B.C. noted "the soul never thinks without an image" (cited in (ibid: 12)) – it also points at the limitation of the present paper as teaching dense mathematical ideas in the spirit of great thinkers of the past requires a "more knowledgeable other" (Vygotsky, 1978) in the classroom.

Three types of technology-immune/technology-enabled (TITE) problems were highlighted. The first type includes problems in which the correctness of computations (TE part) hinges on the correctness of mathematical reasoning (TI part) responsible for the creation of computational algorithms. This type was highlighted through understanding of Figure 7 (section 5) in which the last number in row n is the triangular number of rank $\frac{n(n+1)}{2}$. The second type includes problems in which the diversity of reasoning by induction (TI part) can lead to computing (TE part) the results of which dissent from the context which produced limited numerical evidence to support induction. This type was highlighted through counting the number of binary strings of different lengths by a teacher candidate (section 5). The third type (section 7) includes problems dealing with proving identities (TE part) constructed through the interpretation of geometric constructions (TI part). This type was described as having the correctness of the TI part being dependent on the outcome of the TE part in a sense that the failure to arrive to this outcome indicates the inaccuracy of mathematical thinking involved. A similar idea applies to any use of software in proving relations requiring complex symbolic computations (e.g., relation (6) in section 4).

The activities suggested in the paper can be used primarily with secondary mathematics teacher candidates. At the elementary level of teacher education, some of the activities can be used to demonstrate mathematical ideas presented in Figures 1–3 and Figure 5. At the tertiary level outside of teacher education, the activities can be used with non-mathematics majors as a way of demonstrating the power of the modern-day digital tools which can handle rather sophisticated symbolic computations using simple mathematical algorithms. As it was shown in the paper, despite the relative simplicity of those algorithms, their creation requires certain level of mathematical competency and proficiency in the joint use of reasoning and computations. These skills are important for epistemic and pragmatic development of the 21st century STEM (science technology engineering mathematics)-oriented workforce.

References

Abramovich, S. (2022). Technology-immune/technology-enabled problem solving as agency of design-based mathematics education. *Education Sciences, 12*(8), 514. https://doi.org/10.3390/educsci12080514

Abramovich, S. (2023). Computational triangulation in mathematics teacher education. *Computation, 11*(2), 31. https://doi.org/10.3390/computation11020031

Abramovich, S., Fujii, T., & Wilson, J. W. (1995). Multiple-application medium for the study of polygonal numbers. *Journal of Computers in Mathematics and Science Teaching, 14*(4), 521-558.

Arnheim, R. (1969). *Visual thinking*. Berkeley and Los Angeles, CA: University of California Press.

Asiru, M. A. (2008). A generalization of the formula for the triangular number of the sum and product of natural numbers. *International Journal of Mathematical Education in Science and Technology, 39*(7), 979-985. https://doi.org/10.1080/00207390802136503

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. Retrieved from https://amte.net/standards

Berana, P. J., Montalbo, J., & Magpantay, D. (2015). On triangular and trapezoidal numbers. *Asia Pacific Journal of Multidisciplinary Research, 3*(4), 76-81.

Bütüner, S. Ö. (2016). The use of concrete learning objects taken from the history of mathematics in mathematics education. *International Journal of Mathematical Education in Science and Technology, 47*(8), 1156-1178. https://doi.org/10.1080/0020739x.2016.1184336

Char, B. W., Geddes, K. O., Gonnet, G. H., Leong, B. L., Monagan, M. B., & Watt, S. M. (1995). *Maple V language reference manual*. New York: Springer.

Conference Board of the Mathematical Sciences. (2012). *Mathematical education of teachers II*. Washington, DC: Mathematical Association of America.

Confrey, J. (1995). A theory of intellectual development. *for the Learning of Mathematics, 15*(1), 38-48.

Delvin, K. (1994). *Mathematics: The science of patterns*. New York: Scientific American Library.

Demircioğlu, H. (2023). Preservice mathematics teachers' proving skills in an incorrect statement: Sums of triangular numbers. *Pegem Journal of Education and Instruction*, *13*(1), 326-333. https://doi.org/10.47750/pegegog.13.01.36

Department of Basic Education. (2018). *Mathematics teaching and learning framework for South Africa: Teaching mathematics for understanding*. Private Bag, Pretoria, South Africa: Department of Basic Education.

Dickson, L. E. (2005). *History of the theory of numbers* (Vol. 2). New York: Dover, Diophantine Analysis.

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, *10*(4), 391-408.

Hardy, G. H. (1929). An introduction to the theory of numbers. *Bulletin of the American Mathematical Society*, *35*(6), 778–818.

Hershkowitz, R., Arcavi, A., & Bruckheimer, M. (2001). Reflections on the status and nature of visual reasoning-the case of the matches. *International Journal of Mathematical Education in Science and Technology*, *32*(2), 255-265. https://doi.org/10.1080/00207390010010917

Isoda, M. (2010). *Japanese curriculum standards for mathematics (2012-2020): Junior high school teaching guide for the Japanese course of study: Mathematics (Grade 7-9). Ministry of Education, culture, sports, science and technology (MEXT); CRICED*. Tsukuba, Ibaraki, Japan: University of Tsukuba.

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics*, *71*(3), 235-261. https://doi.org/10.1007/s10649-008-9174-9

Koshy, T. (2002). *Elementary number theory with applications*. New York: Academic Press.

McEwan, H., & Bull, B. (1991). The pedagogic nature of subject matter knowledge. *American Educational Research Journal*, *28*(2), 316-334. https://doi.org/10.2307/1162943

Ministry of Education Singapore. (2020). *Mathematics syllabuses, secondary one to four ministry of education singapore curriculum planning and development division*. Retrieved from https://www.moe.gov.sg/-/media/files/secondary/syllabuses/maths/2020-express_na-maths_syllabuses.pdf?la=en&hash=95B771908EE3D777F87C5D6560EBE6DDAF31D7EF

National Council of Teachers of Mathematics. (2000). *Principles and standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: National Council of Teachers of Mathematics.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core State standards for mathematics*. Washington, DC: Authors.

Nührenbörger, M., Rösken-Winter, B., Fung, C. I., Schwarzkopf, R., Wittmann, E. C., Akinwunmi, K., . . . Schacht, F. (2016). *Design science and its importance in the German mathematics educational discussion Cham*. Switzerland: Springer.

Ontario Ministry of Education. (2020). *The ontario curriculum, grades 1–8, mathematics*. Retrieved from http://www.edu.gov.on.ca

Pedemonte, B., & Buchbinder, O. (2011). Examining the role of examples in proving processes through a cognitive lens: The case of triangular numbers. *ZDM*, *43*, 257-267. https://doi.org/10.1007/s11858-011-0311-z

Plaza, Á. (2016). Proof without words: Sum of triangular numbers. *Mathematics Magazine*, *89*(1), 36-37. https://doi.org/10.2307/27642960

Resnik, M. D. (1999). *Mathematics as a science of patterns*. Oxford, UK: Clarendon Press.

Roegel, D. (2013). *A reconstruction of joncourt's table of triangular numbers (1762) technical report nancy, france: Lorraine laboratory of it research and its applications (a reconstruction of: Elie de joncourt. de natura et præclaro usu simplicissimæ speciei numerorum trigonalium. the hague: husson, 1762)*. Retrieved from http://locomat.loria.fr

Simmt, E., Davis, B., Gordon, L., & Towers, J. (2003). Teachers' mathematics: Curious obligations. *International Group for the Psychology of Mathematics Education*, *4*, 175-182.

Steffe, L. P. (1991). Constructivist teaching experiment in Von Glaserfeld, e. (Ed.), Radical constructivism in mathematics education. In (pp. 177–194). Dordrecht: The Netherlands: Kluwer.

Takahashi, A., Watanabe, T., Yoshida, M., & McDougal, T. (2006). *Lower secondary school teaching guide for the japanese course of study: Mathematics (grade 7-9)*. Madison, NJ: Global Education Resources.

Vavilov, N. A. (2020). Computers as novel mathematical Reality. ii. waring problem. *Computer Tools in Education*, *3*, 5–55.

Vygotsky, L. S. (1930). *The instrumental method in psychology (talk given in 1930 at the krupskaya academy of communist education) lev vygotsky archive*. Retrieved from https://www.marxists.org/archive/vygotsky/works/1930/instrumental.htm

Vygotsky, L. S. (1978). *Mind in society*. Cambridge, MA: Harvard University Press.

Western and Northern Canadian Protocol. (2008). *The common curriculum framework for grades 10–12 mathematics*. Retrieved from http://www.bced.gov.bc.ca/irp/pdfs/mathematics/WNCPmath1012/2008math1012wncp_ccf.pdf

Wing, J. M. (2006). Computational thinking. *Communications of the ACM*, *49*(3), 33–35.

Wittmann, E. C. (2001). Developing mathematics education in a systemic process. *Educational Studies in Mathematics*, *48*(1), 1–20.

Zazkis, R., & Campbell, S. R. (2006). *Number theory in mathematics education: Perspectives and prospects*. Mahwah, NJ: Lawrence Erlbaum.